Kaon Decay into Three Photons Revisited

Shu-Yu Ho and Jusak Tandean

Department of Physics and Center for Theoretical Sciences,
National Taiwan University, Taipei 106, Taiwan

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Abstract

We evaluate the rare radiative kaon decays $K_{\mathrm{L,S}} \to 3\gamma$. Applying the requirements of gauge invariance and Bose symmetry, we derive a general form of the decay amplitude, including both parity-conserving and parity-violating contributions. We employ a chiral-Lagrangian approach combined with dimensional analysis arguments to estimate the branching ratios of these decays in the standard model, obtaining values as large as $\mathcal{B}(K_{\mathrm{L}} \to 3\gamma) \sim 1 \times 10^{-14}$ and $\mathcal{B}(K_{\mathrm{S}} \to 3\gamma) \sim 2 \times 10^{-17}$, which exceed those found previously by a few orders of magnitude. Measurements on the branching ratios which are significantly larger than these numbers would likely hint at the presence of new physics beyond the standard model.

I. INTRODUCTION

The rare kaon decays into three photons, $K_{\rm L} \to 3\gamma$ and $K_{\rm S} \to 3\gamma$, can occur in the absence of CP violation. One might then naively expect from the measured branching ratio $\mathcal{B}\big(K_{\rm L} \to 2\gamma\big) \simeq 5.5 \times 10^{-4}$ [1] that $\mathcal{B}\big(K_{\rm L} \to 3\gamma\big) \sim \alpha_{\rm em}\,\mathcal{B}\big(K_{\rm L} \to 2\gamma\big) \sim 4 \times 10^{-6}$. However, this expectation is already too large in comparison to the result of the first experimental search for the 3γ mode, $\mathcal{B}\big(K_{\rm L} \to 3\gamma\big) < 2.4 \times 10^{-7}$ [1, 2]. As for $K_{\rm S} \to 3\gamma$, there is currently no experimental information available on it, but it is likely to be more suppressed than expected as well.

It turns out that the considerable smallness of the $K \to 3\gamma$ rates has to do with the constraints imposed on the decay amplitude by gauge invariance and Bose symmetry [3]. Gauge invariance implies that the total angular momentum J of any two of the three photons in the 3γ final-state cannot be zero, whereas Bose statistics forbids the photon pair to have $J = 1$. Since each of the photon pairs must have $J \geq 2$, the decay amplitude suffers from a large number of angular momentum suppression factors.

The rates of $K_{\rm L,S} \to 3\gamma$ were roughly estimated a while ago in Ref. [3]. The calculation was based on a simple model in which $K \to 3\gamma$ is assumed to proceed from $K \to \pi^0\pi^0\gamma$ with the π^0 pair immediately converting into a photon pair. The resulting branching ratios are tiny, $\mathcal{B}\big(K_{\rm L} \to 3\gamma\big) \sim 3 \times 10^{-19}$ and $\mathcal{B}\big(K_{\rm S} \to 3\gamma\big) \sim 5 \times 10^{-22}$ [3].

Here we take another look at these rare decays, partly motivated by a new search for $K_{\rm L} \to 3\gamma$ currently being performed in the E391a experiment at KEK [4]. Since the estimate obtained in Ref. [3] resulted from only one contributing diagram, it is possible that other contributions exist which can enhance the decay rate. In this study we start with a general form of the $K \to 3\gamma$ amplitude subject to the restrictions from gauge invariance and Bose statistics. We then use a chiral-Lagrangian framework along with dimensional-analysis arguments to explore the size of various important contributions. This finally leads us to $K_{\rm L,S} \to 3\gamma$ rates which exceed those previously estimated by a few orders of magnitude.

II. DECAY AMPLITUDES AND RATES

The decay $K \to 3\gamma$ being a weak transition, its amplitude $\mathcal{M}(K \to 3\gamma)$ generally consists of separate terms describing the parity conserving (PC) and parity violating (PV) components of the transition. Accordingly, one can write $\mathcal{M}(K \to 3\gamma)$ as a sum of their respective contributions,

$$\mathcal{M}(K \to 3\gamma) \;=\; \mathcal{M}_{\rm PC}^{K} + \mathcal{M}_{\rm PV}^{K} \,, \tag{1}$$

$$\mathcal{M}_{\rm PC}^{K} \;=\; \varepsilon_{1\alpha}^{*}\varepsilon_{2\beta}^{*}\varepsilon_{3\mu}^{*}\, M_{\rm PC}^{\alpha\beta\mu} \,, \qquad \mathcal{M}_{\rm PV}^{K} \;=\; \varepsilon_{1\alpha}^{*}\varepsilon_{2\beta}^{*}\varepsilon_{3\mu}^{*}\, M_{\rm PV}^{\alpha\beta\mu} \,, \tag{2}$$

where $\varepsilon_{1,2,3}$ are the polarization vectors of the photons.

To construct a general form of the $K \to 3\gamma$ amplitude, one follows a well-known prescription. The presence of photons implies that $\mathcal{M}(K \to 3\gamma)$ has to be gauge invariant. With three photons in the final state, Bose statistics dictates that the amplitude be symmetric under interchange of any two of the photons. These requirements must be satisfied by $\mathcal{M}_{\rm PC}$ and $\mathcal{M}_{\rm PV}$ separately.

We deal with $M_{\rm PV}$ first, as it turns out to be simpler than $M_{\rm PC}$ and is also more relevant to $K_{\rm L} \to 3\gamma$, which is our main decay of interest, relegating some details to Appendix A. We also assume that the photons are on-shell. Thus we get

$$\begin{aligned} M_{\rm PV}^{\alpha\beta\mu} \;=\;& \big(g^{\alpha\beta} z - k_2^\alpha k_1^\beta\big)\big(k_1^\mu\, y - k_2^\mu\, x\big)\, F(x,y,z) \\ &+ \big(g^{\beta\mu} y - k_3^\beta k_2^\mu\big)\big(k_2^\alpha\, x - k_3^\alpha\, z\big)\, F(z,x,y) \\ &+ \big(g^{\alpha\mu} x - k_3^\alpha k_1^\mu\big)\big(k_3^\beta\, z - k_1^\beta\, y\big)\, F(y,z,x) \\ &+ \big[g^{\alpha\beta}\,\big(k_1^\mu\, y - k_2^\mu\, x\big) + g^{\beta\mu}\,\big(k_2^\alpha\, x - k_3^\alpha\, z\big) \\ &\quad + g^{\alpha\mu}\,\big(k_3^\beta\, z - k_1^\beta\, y\big) \,+\, k_3^\alpha k_1^\beta k_2^\mu - k_2^\alpha k_3^\beta k_1^\mu\big]\, G(x,y,z) \;, \end{aligned} \tag{3}$$

where $k_{1,2,3}$ are the momenta of the photons with polarizations $\varepsilon_{1,2,3}$, respectively,

$$x \;=\; k_1\cdot k_3 \;, \qquad y \;=\; k_2\cdot k_3 \;, \qquad z \;=\; k_1\cdot k_2 \;, \tag{4}$$

and the functions F and G must be free of kinematic singularities and satisfy the relations

$$\begin{aligned} F(u,v,w) \;&=\; -F(v,u,w) \;, \\ G(u,v,w) \;=\; -G(v,u,w) \;&=\; -G(w,v,u) \;=\; -G(u,w,v) \;, \end{aligned} \tag{5}$$

with u,v,w each being any one of the invariants $k_i\cdot k_j$. This amplitude agrees with the one derived in Ref. [5] for $\pi^0 \to 3\gamma$.

For the parity-conserving contribution, the form with the desired symmetry properties can be expressed as

$$\begin{aligned} M_{\rm PC}^{\alpha\beta\mu} \;=\;& \Big[\big(g^{\alpha\beta} z - k_2^\alpha k_1^\beta\big)\,\epsilon^{\mu\rho\sigma\tau}\,\mathcal{F}(x,y,z) + \big(g^{\beta\mu} y - k_3^\beta k_2^\mu\big)\,\epsilon^{\alpha\rho\sigma\tau}\,\mathcal{F}(z,x,y) \\ &\quad + \big(g^{\alpha\mu} x - k_3^\alpha k_1^\mu\big)\,\epsilon^{\beta\rho\sigma\tau}\,\mathcal{F}(y,z,x)\Big] k_{1\rho} k_{2\sigma} k_{3\tau} \\ &+ \Big[\big(k_2^\mu k_1^\tau - k_1^\mu k_2^\tau\big)\,\epsilon^{\alpha\beta\rho\sigma}\,\mathcal{H}(x,y,z) + \big(k_3^\alpha k_2^\rho - k_2^\alpha k_3^\rho\big)\,\epsilon^{\beta\mu\sigma\tau}\,\mathcal{H}(z,x,y) \\ &\quad + \big(k_1^\beta k_3^\sigma - k_3^\beta k_1^\sigma\big)\,\epsilon^{\alpha\mu\rho\tau}\,\mathcal{H}(y,z,x)\Big] k_{1\rho} k_{2\sigma} k_{3\tau} \\ &+ \tfrac{1}{3}\big(g^{\alpha\beta}\epsilon^{\mu\rho\sigma\tau} + g^{\rho\sigma}\epsilon^{\alpha\beta\mu\tau} + g^{\beta\rho}\epsilon^{\alpha\mu\sigma\tau} - g^{\alpha\sigma}\epsilon^{\beta\mu\rho\tau} \\ &\quad + g^{\beta\mu}\epsilon^{\alpha\rho\sigma\tau} + g^{\sigma\tau}\epsilon^{\alpha\beta\mu\rho} + g^{\mu\sigma}\epsilon^{\alpha\beta\rho\tau} - g^{\beta\tau}\epsilon^{\alpha\mu\rho\sigma} \\ &\quad + g^{\alpha\mu}\epsilon^{\beta\rho\sigma\tau} + g^{\rho\tau}\epsilon^{\alpha\beta\mu\sigma} + g^{\mu\rho}\epsilon^{\alpha\beta\sigma\tau} - g^{\alpha\tau}\epsilon^{\beta\mu\rho\sigma}\big) k_{1\rho} k_{2\sigma} k_{3\tau}\, \mathcal{G}(x,y,z) \;, \end{aligned} \tag{6}$$

where the functions $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ are also free of kinematic singularities and satisfy

$$\begin{aligned} \mathcal{F}(u,v,w) \;=\; -\mathcal{F}(v,u,w) \;, \qquad\qquad & \mathcal{H}(u,v,w) \;=\; -\mathcal{H}(v,u,w) \;, \\ \mathcal{G}(u,v,w) \;=\; -\mathcal{G}(v,u,w) \;=\; -\mathcal{G}&(w,v,u) \;=\; -\mathcal{G}(u,w,v) \;. \end{aligned} \tag{7}$$

The formula for $M_{\rm PC}$ above may have been constructed for the first time in this paper.

After summing $|\mathcal{M}_{\rm PV}^K + \mathcal{M}_{\rm PC}^K|^2$ over the photon polarizations, we find that there is no interference between the PC and PV contributions in the result, in accord with expectation. It is given by

$$\sum_{\rm pol} |\mathcal{M}(K\to 3\gamma)|^2 \;=\; \sum_{\rm pol}\Big(\big|\mathcal{M}_{\rm PV}^K\big|^2 \,+\, \big|\mathcal{M}_{\rm PC}^K\big|^2\Big) \;, \tag{8}$$

where

$$\sum_{\rm pol}\left|\mathcal{M}_{\rm PV}^K\right|^2 \;=\; 4\big\{|F_1|^2z^2+|F_2|^2y^2+|F_3|^2x^2\;+\;2\,|G|^2 \\ +\;{\rm Re}\big[F_1^*F_2\,y\,z+F_2^*F_3\,x\,y+F_3^*F_1\,x\,z+2\big(F_1^*z+F_2^*y+F_3^*x\big)G\big]\big\}x\,y\,z\;, \tag{9}$$

$$\sum_{\rm pol}\left|\mathcal{M}_{\rm PC}^K\right|^2 \;=\; 4\big\{\big(|\mathcal{F}_1|^2+|\mathcal{H}_1|^2\big)z^2+\big(|\mathcal{F}_2|^2+|\mathcal{H}_2|^2\big)y^2+\big(|\mathcal{F}_3|^2+|\mathcal{H}_3|^2\big)x^2\;+\;2\,|\mathcal{G}|^2 \\ +\;{\rm Re}\big[\big(\mathcal{F}_1^*+\mathcal{H}_1^*\big)\big(\mathcal{F}_2+\mathcal{H}_2+2\mathcal{G}/y\big)y\,z+\big(\mathcal{F}_2^*+\mathcal{H}_2^*\big)\big(\mathcal{F}_3+\mathcal{H}_3+2\mathcal{G}/x\big)x\,y \\ +\;\big(\mathcal{F}_3^*+\mathcal{H}_3^*\big)\big(\mathcal{F}_1+\mathcal{H}_1+2\mathcal{G}/z\big)x\,z\big]\big\}x\,y\,z\;, \tag{10}$$

$$F_1\;=\;F(x,y,z)\;,\qquad F_2\;=\;F(z,x,y)\;,\qquad F_3\;=\;F(y,z,x)\;, \tag{11}$$

similarly for $\mathcal{F}_{1,2,3}$ and $\mathcal{H}_{1,2,3}$, and

$$G\;=\;G(x,y,z)\;,\qquad \mathcal{G}\;=\;\mathcal{G}(x,y,z)\;. \tag{12}$$

The resulting decay rate can be written as

$$\Gamma(K\to 3\gamma)\;=\;\frac{1}{256\,\pi^3\,m_K^3}\,\frac{1}{3!}\int ds_{12}\,ds_{23}\sum_{\rm pol}|\mathcal{M}(K\to 3\gamma)|^2\;, \tag{13}$$

where the 3! accounts for the three photons being identical particles and $s_{mn}=(k_m+k_n)^2$. Using the formulas above, we provide our numerical estimates in the next section.

Before moving on, we remark that the expressions in Eqs. (3), (6), (9), and (10) apply more generally to the decay of other neutral pseudoscalar particle into three photons. Furthermore, they also work for a neutral scalar particle decaying into three photons, but with the PC and PV contributions interchanged.

III. ESTIMATE OF $K_{\rm L,S}\to 3\gamma$ RATES

To explore the size of the leading contributions to $\mathcal{M}(K\to 3\gamma)$, we adopt a chiral-Lagrangian approach [6]. Accordingly, they are expected to arise from the relevant terms in the chiral expansion and give rise to terms in the functions F, G, $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ with the lowest numbers of powers of the photon momenta k_i. Since there are in principle many possible contributions to the amplitude, from tree and loop diagrams, with mostly unknown parameters, we consider a few representative contributions and rely on dimensional-analysis arguments to determine their size.

A. $K_{\rm L}\to 3\gamma$

Neglecting CP violation, we can concentrate on the PV part of the amplitude, in Eq. (3), as $K_{\rm L}\to 3\gamma$ violates charge-conjugation invariance. Thus, for F and G satisfying Eq. (5) we find the simplest form

$$F(u,v,w)\;=\;c_F(u-v)\;, \\ G(u,v,w)\;=\;c_G[(u-v)f(w)+(v-w)f(u)+(w-u)f(v)]\;, \tag{14}$$

where $c_{F,G}$ are constants and f is any well-behaved function, although it cannot be a constant if G is to be nonzero. This implies that F and G contain at least two and four powers of k_i, respectively, as the momentum power in the chiral expansion in the meson sector always increases by even numbers. It follows that $M_{\rm PV}$ in Eq. (3) contains at least seven powers of k_i.

To assess the leading contributions to $M_{\rm PV}$, we first consider an example of a weak chiral Lagrangian for strangeness changing, $|\Delta S|=1$, transitions within the standard model (SM) which is odd under parity, has seven derivatives, and couples a kaon to three photons in a gauge-invariant way. As is well known, the weak chiral Lagrangian for such transitions in the SM is dominated by contributions which transform as $\left(8_{\rm L},1_{\rm R}\right)$ [6] and has to be invariant under the CPS transformation [7], which is the product of the ordinary CP transformation and the switching of the s and d quarks. An example with the required properties is

$$\begin{aligned}\mathcal{L}_{\rm PV} &= c_7\left\langle \xi^\dagger h\xi\left(\nabla^\alpha\mathcal{V}^{\mu\nu}\right)\left[\mathcal{U}^\rho\,\nabla_\alpha\mathcal{V}_{\rho\sigma}+\left(\nabla_\sigma\mathcal{V}_{\rho\alpha}\right)\mathcal{U}^\rho\right]\nabla^\sigma\mathcal{V}_{\mu\nu}\right\rangle + \text{H.c.} \\ &= \frac{8\sqrt{2}\,c_7\,e^3}{27\,f_\pi}\,\partial^\alpha F^{\mu\nu}\left(\partial_\alpha F_{\rho\sigma}+\partial_\sigma F_{\rho\alpha}\right)\partial^\rho\bar{K}^0\,\partial^\sigma F_{\mu\nu} + \cdots + \text{H.c.}\,,\end{aligned} \tag{15}$$

where c_7 is a constant, $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ is the usual photon field strength tensor, and only the relevant part is displayed in the second line. In the first line, $\xi=e^{i\varphi/(2f)}$ contains the lightest octet of pseudoscalar mesons via the 3×3 matrix φ [6], with $f=f_\pi=92\,\text{MeV}$ being the pion decay constant, h is a 3×3 matrix having elements $h_{kl}=\delta_{k2}\delta_{3l}$ which selects out $s\to d$ transitions, $\mathcal{V}_{\mu\nu}=e\left(\xi^\dagger Q\xi+\xi Q\xi^\dagger\right)F_{\mu\nu}$ with $Q=\text{diag}(2,-1,-1)/3$ being the quark charge matrix, $\nabla_\alpha X=\partial_\alpha X+\frac{1}{2}\left[\xi^\dagger\,\partial_\alpha\xi+\xi\,\partial_\alpha\xi^\dagger+ie\left(\xi Q\xi^\dagger+\xi^\dagger Q\xi\right)A_\alpha,X\right]$, and $\mathcal{U}_\alpha=i\left(\xi^\dagger\,\partial_\alpha\xi-\xi\,\partial_\alpha\xi^\dagger\right)+e\left(\xi Q\xi^\dagger-\xi^\dagger Q\xi\right)A_\alpha$. The Lagrangian in Eq. (15) yields

$$F(u,v,w) = \frac{32\sqrt{2}\,ic_7\,e^3}{27\,f_\pi}(u-v)\,, \qquad G(u,v,w) = 0 \tag{16}$$

in $\mathcal{M}\left(\bar{K}^0\right)$ and the same functions in $\mathcal{M}\left(K^0\right)$, but with c_7 replaced with c_7^* if CP violation is present.

If CP is conserved, only $\mathcal{M}_{\rm PV}$ contributes to the $K_{\rm L}\to 3\gamma$ amplitude. In that case, upon applying Eq. (16) in Eq. (9) and adopting the convention $K_{\rm L}=\left(K^0+\bar{K}^0\right)/\sqrt{2}$, we find

$$\sum_{\rm pol}\left|\mathcal{M}\left(K_{\rm L}\to 3\gamma\right)\right|^2 = \frac{|128\,c_7|^2\,e^6}{729\,f_\pi^2}\left(x^2y^2+y^2z^2+x^2z^2-x\,y\,z^2-x\,y^2\,z-x^2\,y\,z\right)x\,y\,z\,, \tag{17}$$

having made use of Eq. (4). To calculate the decay rate, we then need the value of c_7. Since it is not possible at present to determine this constant rigorously from the quark-level parameters, we estimate it with the aid of naive dimensional analysis [8]. Thus we obtain the order-of-magnitude value

$$c_7 \sim \frac{G_{\rm F}\,\lambda_{\rm C}\,f_\pi^4}{\sqrt{2}\,\Lambda^8} \simeq 1.0\times 10^{-9}\ \text{GeV}^{-6}\,, \tag{18}$$

where $\lambda_{\rm C}=0.22$ is the Cabibbo mixing parameter and Λ represents the scale at which the chiral Lagrangian approach breaks down, and so we take $\Lambda=m_\rho=775\,\text{MeV}$ [1]. The resulting branching ratio is

$$\mathcal{B}\left(K_{\rm L}\to 3\gamma\right) \sim 7.4\times 10^{-17}\,. \tag{19}$$

This number is about 250 times larger than the earlier prediction, $\mathcal{B}\big(K_{\rm L}\to 3\gamma\big)\sim 3\times 10^{-19}$, given in Ref. [3]. We have repeated that calculation and come to a different result, which we briefly discuss here. In the model employed in Ref. [3], the amplitude $\mathcal{M}\big(K_{\rm L}\to 3\gamma\big)$ is roughly approximated by

$$\begin{aligned}
{\rm Im}\,\mathcal{M}\big(K_{\rm L}\to 3\gamma\big) \;=\; & \theta\big(s_{12}-4m_\pi^2\big)\,\tfrac{1}{2}\int\frac{d^3p_1}{(2\pi)^3\,2p_{10}}\frac{d^3p_2}{(2\pi)^3\,2p_{20}}\,(2\pi)^4\,\delta^{(4)}\big(p_K-p_1-p_2-k_3\big) \\
& \times\frac{h_{\rm L}}{m_K^5}\big(\varepsilon_3^*\!\cdot p_1\,k_3\!\cdot p_K-\varepsilon_3^*\!\cdot p_K\,k_3\!\cdot p_1\big)\,k_3\!\cdot\big(2p_1-p_K\big) \\
& \times\frac{\tilde{G}\,s_{12}}{m_\rho^2}\left(\frac{k_1\!\cdot p_1\,k_2\!\cdot p_1}{k_1\!\cdot k_2}\,g_{\mu\nu}+p_{1\mu}p_{1\nu}-\frac{k_1\!\cdot p_1}{k_1\!\cdot k_2}\,k_{2\mu}p_{1\nu}-\frac{k_2\!\cdot p_1}{k_1\!\cdot k_2}\,k_{1\nu}p_{1\mu}\right)\varepsilon_1^{*\mu}\varepsilon_2^{*\nu} \\
& +\big[\text{permutations of }\big(\varepsilon_1,k_1\big),\,\big(\varepsilon_2,k_2\big),\,\big(\varepsilon_3,k_3\big)\big]\;,
\end{aligned}\qquad(20)$$

coming from a π^0-loop diagram, where p_K is the $K_{\rm L}$ momentum, $p_{1,2}$ are the π^0 momenta in the loop, $h_{\rm L}\simeq 6.6\times 10^{-8}$, and $\tilde{G}=\frac{10}{9}\,g_{\omega\pi\gamma}^2$ with $g_{\omega\pi\gamma}=0.77\,{\rm GeV}^{-1}$. To compute this requires the evaluation of the integrals

$$\begin{aligned}
K^{\mu\nu\rho\sigma} &= \int\frac{d^3p_1}{2p_{10}}\frac{d^3p_2}{2p_{20}}\,\delta^{(4)}\big(P-p_1-p_2\big)\,p_1^\mu p_1^\nu p_1^\rho p_1^\sigma\,f\big(p_1\cdot p_2\big)\;, \\
L^{\mu\nu\rho\sigma} &= \int\frac{d^3p_1}{2p_{10}}\frac{d^3p_2}{2p_{20}}\,\delta^{(4)}\big(P-p_1-p_2\big)\,p_1^\mu p_1^\nu p_1^\rho p_2^\sigma\,f\big(p_1\cdot p_2\big)\;.
\end{aligned}\qquad(21)$$

We have collected the results in Appendix B, where the expression for $K^{\mu\nu\rho\sigma}$ agrees with that given in Ref. [3], but our $L^{\mu\nu\rho\sigma}$ differs from theirs. With ${\rm Im}\,\mathcal{M}\big(K_{\rm L}\to 3\gamma\big)$ computed and subsequently equated to $\mathcal{M}_{\rm PV}\big(K_{\rm L}\big)=\varepsilon_{1\alpha}^*\varepsilon_{2\beta}^*\varepsilon_{3\mu}^*M_{\rm PV}^{\alpha\beta\mu}$, we then extract

$$\begin{aligned}
F(u,v,w) &= c\left[\frac{1}{\sqrt{v}}\big(v-2m_\pi^2\big)^{5/2}\,\theta\big(v-2m_\pi^2\big)\;-\;\frac{1}{\sqrt{u}}\big(u-2m_\pi^2\big)^{5/2}\,\theta\big(u-2m_\pi^2\big)\right]\;, \\
G(u,v,w) &= \frac{c}{2}\left[\frac{w-v}{\sqrt{u}}\big(u-2m_\pi^2\big)^{5/2}\;\theta\big(u-2m_\pi^2\big)\;+\;\frac{u-w}{\sqrt{v}}\big(v-2m_\pi^2\big)^{5/2}\,\theta\big(v-2m_\pi^2\big)\right. \\
&\qquad\left.+\;\frac{v-u}{\sqrt{w}}\big(w-2m_\pi^2\big)^{5/2}\,\theta\big(w-2m_\pi^2\big)\right]\;,
\end{aligned}\qquad(22)$$

$$c \;=\; \frac{\tilde{G}\,h_{\rm L}}{120\,\pi\,m_K^5\,m_\rho^2}\;\simeq\;6.3\times 10^{-9}\ {\rm GeV}^{-9}\;. \qquad(23)$$

Incorporating these into Eqs. (9) and (13) leads us to $\mathcal{B}\big(K_{\rm L}\to 3\gamma\big)\sim 1.0\times 10^{-18}$, which is three times greater than the number found in Ref. [3].

Since the results in the preceding paragraph arise from a loop diagram [3], it is instructive to compare them to tree-level contributions of the same chiral order. From the F and G formulas in Eq. (22) we can see that they contain four and six powers of the photon momenta k_i, respectively, and hence their contributions to $\mathcal{M}\big(K_{\rm L}\to 3\gamma\big)$ have nine powers of k_i. If a weak chiral Lagrangian with nine derivatives contributes to this amplitude at tree level, the size of its coupling constant in $F(u,v,w)$ is predicted by naive dimensional analysis to be

$$c'\;\sim\;\frac{e^3\,G_{\rm F}\,\lambda_{\rm C}\,f_\pi^3}{\sqrt{2}\,\Lambda^{10}}\;\sim\;5.0\times 10^{-10}\ {\rm GeV}^{-9}\;. \qquad(24)$$

This is about 12 times smaller than c in Eq. (23). One may then suggest based on this comparison that loop contributions with seven powers of k_i could also be enhanced by a similar amount relative to the tree-level contribution from Lagrangians with seven derivatives, such as that in Eq. (15). If such enhancement occurs, we may have

$$\mathcal{B}(K_{\rm L} \to 3\gamma) \;\sim\; 1 \times 10^{-14} \,, \tag{25}$$

instead of Eq. (19). The numbers in Eqs. (19) and (25) can be taken to be representative values coming from the various contributions to this decay and also to indicate the level of uncertainty involved in our crude calculation. It is therefore reasonable to conclude that

$$7 \times 10^{-17} \;\lesssim\; \mathcal{B}(K_{\rm L} \to 3\gamma) \;\lesssim\; 1 \times 10^{-14} \,. \tag{26}$$

If CP violation is not neglected, then $\mathcal{M}_{\rm PC}^K$ also contributes to the $K_{\rm L} \to 3\gamma$ rate, via the ϵ_K term in $K_{\rm L} \simeq \left[K^0 + \bar{K}^0 + \epsilon_K\left(K^0 - \bar{K}^0\right)\right]/\sqrt{2}$, where $\epsilon_K \sim 2 \times 10^{-3}$ is the CP-violation parameter in kaon mixing. Now, the similarity between Eqs. (5) and (7) suggests that $\mathcal{F}$ and $\mathcal{H}$ ($\mathcal{G}$) are comparable in size to F (G). We can then expect that $\mathcal{M}_{\rm PC}^K$ is, at most, also comparable to $\mathcal{M}_{\rm PV}^K$. This implies that the effect of CP violation on $\mathcal{B}(K_{\rm L} \to 3\gamma)$ is small, and hence the predicted branching ratio is still what is quoted in Eq. (26).

B. $K_{\rm S} \to 3\gamma$

The rate of this decay is determined mostly by the parity-conserving contribution $\mathcal{M}_{\rm PC}^K$. The branching ratio predicted in Ref. [3] is $\mathcal{B}(K_{\rm S} \to 3\gamma) \sim 5 \times 10^{-22}$. Repeating the calculation, as in the $K_{\rm L} \to 3\gamma$ case, we find instead a value three times larger, $\mathcal{B}(K_{\rm S} \to 3\gamma) \sim 1.8 \times 10^{-21}$. Since this arises from a loop contribution involving nine powers of the photon momenta k_i, we need to consider as before the lower-order contributions, with seven powers of k_i, which may be larger.

We take the leading-order form $\mathcal{F}(u,v,w) \sim \mathcal{H}(u,v,w) = \tilde{c}(u-v)$, satisfying Eq. (7), with $\tilde{c}$ being a constant and $\mathcal{G} = 0$. In this case, the situation is similar to that of $\mathcal{M}_{\rm PC}^K$ with F and G given in Eq. (16). More precisely, making a comparison of $\sum_{\rm pol}|\mathcal{M}_{\rm PC}^K|^2$ in Eq. (10) and $\sum_{\rm pol}|\mathcal{M}_{\rm PV}^K|^2$ in Eq. (9) for the two cases, respectively, one can see that their decay distributions have the same functional dependence on x, y, and z. It follows that $\Gamma(K_{\rm S} \to 3\gamma)$ can be expected to be roughly of the same order as $\Gamma(K_{\rm L} \to 3\gamma)$. Interestingly, the measured rates of the corresponding 2γ modes are also of similar order, $\Gamma(K_{\rm S} \to 2\gamma) \sim 2.7\,\Gamma(K_{\rm L} \to 2\gamma)$ [1]. In view of Eq. (26), we can therefore predict that

$$1 \times 10^{-19} \;\lesssim\; \mathcal{B}(K_{\rm S} \to 3\gamma) \;\lesssim\; 2 \times 10^{-17} \,. \tag{27}$$

IV. CONCLUSIONS

We have revisited the rare kaon decay $K \to 3\gamma$, which is expected to be much suppressed because its amplitude has a large number of angular momentum suppression factors. We have

constructed a general form of the decay amplitude which satisfies the requirements of gauge invariance and Bose symmetry and includes both parity-conserving and parity-violating contributions. We have in addition calculated the squared amplitude, summed over the photon polarizations, which can be useful to produce a Dalitz plot distribution of the decay. These results are applicable generally to the decay of any spinless particle into three photons.

More specifically, we have dealt mainly with $K_{\rm L}\to 3\gamma,$ which is currently the subject of a new experimental search at KEK, but also evaluated $K_{\rm S}\to 3\gamma,$ albeit more briefly. To explore the leading-order contributions to their amplitudes, we have adopted a chiral-Lagrangian approach in the context of the standard model. This implies that there are many possible contributions to the amplitudes, from tree and loop diagrams, with mostly unknown parameters. Consequently, for definiteness we have considered a number of representative contributions and used dimensional-analysis arguments to estimate their size. This has finally led us to arrive at branching ratios as large as $\mathcal{B}\bigl(K_{\rm L}\to 3\gamma\bigr)\sim 1\times 10^{-14}$ and $\mathcal{B}\bigl(K_{\rm S}\to 3\gamma\bigr)\sim 2\times 10^{-17},$ which exceed those estimated before by a few orders of magnitude, but are still very small. Nevertheless, any experimental findings on the branching ratios which are significantly greater than these numbers would likely signal the effect of new physics beyond the standard model.

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Appendix A: Derivation of amplitudes

To obtain a general form of the $K\to 3\gamma$ amplitude with the desired properties, we start with expressions for $M_{\rm PV}^{\alpha\beta\mu}$ and $M_{\rm PC}^{\alpha\beta\mu}$ consisting of terms involving all possible combinations of the available tensors, $k_{1,2,3}^\alpha$, $g^{\eta\kappa}$, and $\epsilon^{\mu\nu\sigma\tau}$. Thus we have

$$\begin{aligned} M_{\rm PV}^{\alpha\beta\mu} \;=\;& g^{\alpha\beta}\left(k_1^\mu\,a_1+k_2^\mu\,a_2\right)+g^{\alpha\mu}\left(k_1^\beta\,a_3+k_3^\beta\,a_4\right)+g^{\beta\mu}\left(k_2^\alpha\,a_5+k_3^\alpha\,a_6\right) \\ &+k_2^\alpha\bigl[k_1^\beta\bigl(k_1^\mu b_1+k_2^\mu b_2\bigr)+k_3^\beta\bigl(k_1^\mu b_3+k_2^\mu b_4\bigr)\bigr]+k_3^\alpha\bigl[k_1^\beta\bigl(k_1^\mu b_5+k_2^\mu b_6\bigr)+k_3^\beta\bigl(k_1^\mu b_7+k_2^\mu b_8\bigr)\bigr]\;, \end{aligned} \tag{A1}$$

$$\begin{aligned} M_{\rm PC}^{\alpha\beta\mu} \;=\;& \epsilon^{\alpha\beta\mu\rho}\left(k_{1\rho}c_1+k_{2\rho}c_2+k_{3\rho}c_3\right)+\bigl(g^{\alpha\beta}\epsilon^{\mu\rho\sigma\tau}d_1+g^{\alpha\mu}\epsilon^{\beta\rho\sigma\tau}d_2+g^{\beta\mu}\epsilon^{\alpha\rho\sigma\tau}d_3\bigr)k_{1\rho}k_{2\sigma}k_{3\tau} \\ &+\bigl[\bigl(k_2^\alpha\,f_1+k_3^\alpha\,f_2\bigr)\epsilon^{\beta\mu\rho\sigma}+\bigl(k_1^\beta\,f_3+k_3^\beta\,f_4\bigr)\epsilon^{\alpha\mu\rho\sigma}+\bigl(k_1^\mu\,f_5+k_2^\mu\,f_6\bigr)\epsilon^{\alpha\beta\rho\sigma}\bigr]k_{1\rho}k_{2\sigma} \\ &+\bigl[\bigl(k_3^\alpha\,g_1+k_2^\alpha\,g_2\bigr)\epsilon^{\beta\mu\rho\sigma}+\bigl(k_1^\mu\,g_3+k_2^\mu\,g_4\bigr)\epsilon^{\alpha\beta\rho\sigma}+\bigl(k_1^\beta\,g_5+k_3^\beta\,g_6\bigr)\epsilon^{\alpha\mu\rho\sigma}\bigr]k_{1\rho}k_{3\sigma} \\ &+\bigl[\bigl(k_2^\mu\,h_1+k_1^\mu\,h_2\bigr)\epsilon^{\alpha\beta\rho\sigma}+\bigl(k_3^\beta\,h_3+k_1^\beta\,h_4\bigr)\epsilon^{\alpha\mu\rho\sigma}+\bigl(k_3^\alpha\,h_5+k_2^\alpha\,h_6\bigr)\epsilon^{\beta\mu\rho\sigma}\bigr]k_{2\rho}k_{3\sigma} \\ &+\bigl\{\bigl[k_2^\alpha\bigl(k_1^\beta l_1+k_3^\beta l_2\bigr)+k_3^\alpha\bigl(k_1^\beta l_3+k_3^\beta l_4\bigr)\bigr]\epsilon^{\mu\rho\sigma\tau}+\bigl[k_2^\alpha\bigl(k_1^\mu l_5+k_2^\mu l_6\bigr)+k_3^\alpha\bigl(k_1^\mu l_7+k_2^\mu l_8\bigr)\bigr]\epsilon^{\beta\rho\sigma\tau} \\ &\quad+\bigl[k_1^\beta\bigl(k_1^\mu l_9+k_2^\mu l_{10}\bigr)+k_3^\beta\bigl(k_1^\mu l_{11}+k_2^\mu l_{12}\bigr)\bigr]\epsilon^{\alpha\rho\sigma\tau}\bigr\}k_{1\rho}k_{2\sigma}k_{3\tau}\;, \end{aligned} \tag{A2}$$

where the a_i, b_i, c_i, d_i, f_i, g_i, h_i, and l_i are functions dependent on the invariants $k_i\cdot k_j$. After the requirements of gauge invariance and Bose symmetry have been imposed on $M_{\rm PV}^{\alpha\beta\mu}$ and $M_{\rm PC}^{\alpha\beta\mu}$

separately, they each contain a much smaller number of functions. For $M^{\alpha\beta\mu}_{\rm PC}$, we make further simplification with the aid of Schouten's identity, which states that in four dimensions a tensor with five or more Lorentz indices vanishes identically if it is completely antisymmetric with respect to five or more of the indices.[1] Such a tensor is

$$g^{\alpha\mu}\epsilon^{\nu\rho\sigma\tau} - g^{\alpha\nu}\epsilon^{\mu\rho\sigma\tau} - g^{\alpha\rho}\epsilon^{\nu\mu\sigma\tau} - g^{\alpha\sigma}\epsilon^{\nu\rho\mu\tau} - g^{\alpha\tau}\epsilon^{\nu\rho\sigma\mu} \;=\; 0 \;, \tag{A3}$$

which is fully antisymmetric with respect to μ,ν,ρ,σ,τ. We display our results for $M_{\rm PV}$ and $M_{\rm PC}$ in Eqs. (3) and (6), respectively, in the case of on-shell photons. The $\mathcal{G}$ terms in Eq. (6) could be simplified further using Eq. (A3), but then they would not be manifestly Bose-symmetric.

Appendix B: Integrals

The integrals in Eq. (21) can be written in terms of all the possible appropriate combinations of the available tensors, $g^{\alpha\beta}$ and P^η, as

$$\begin{aligned} K^{\mu\nu\rho\sigma} \;=\; \frac{\pi\,\lambda^{\frac{1}{2}}\big(s,m_1^2,m_2^2\big)}{10\,s}\Big[& \big(g^{\mu\nu}g^{\rho\sigma}+g^{\mu\rho}g^{\nu\sigma}+g^{\mu\sigma}g^{\nu\rho}\big)K_1 + P^\mu P^\nu P^\rho P^\sigma\,K_3 \\ & + \big(g^{\mu\nu}P^\rho P^\sigma + g^{\mu\rho}P^\nu P^\sigma + g^{\mu\sigma}P^\nu P^\rho + g^{\nu\rho}P^\mu P^\sigma \\ & \quad + g^{\nu\sigma}P^\mu P^\rho + g^{\rho\sigma}P^\mu P^\nu\big)K_2\Big]\, f\Big(\tfrac{1}{2}\big(s-m_1^2-m_2^2\big)\Big) \;, \end{aligned} \tag{B1}$$

$$\begin{aligned} L^{\mu\nu\rho\sigma} \;=\; \frac{\pi\,\lambda^{\frac{1}{2}}\big(s,m_1^2,m_2^2\big)}{10\,s}\Big[& \big(g^{\mu\nu}g^{\rho\sigma}+g^{\mu\rho}g^{\nu\sigma}+g^{\nu\rho}g^{\mu\sigma}\big)L_1 + P^\mu P^\nu P^\rho P^\sigma\,L_4 \\ & + \big(g^{\mu\nu}P^\rho P^\sigma + g^{\mu\rho}P^\nu P^\sigma + g^{\nu\rho}P^\nu P^\sigma\big)L_2 \\ & + \big(g^{\mu\sigma}P^\nu P^\rho + g^{\nu\sigma}P^\mu P^\rho + g^{\rho\sigma}P^\mu P^\nu\big)L_3\Big]\, f\Big(\tfrac{1}{2}\big(s-m_1^2-m_2^2\big)\Big) \;, \end{aligned} \tag{B2}$$

where $\lambda(u,v,w) = u^2+v^2+w^2-2u\,v-2v\,w-2u\,w$, $m_1^2=p_1^2$, $m_2^2=p_2^2$, and the coefficients $K_{1,2,3}$ and $L_{1,2,3,4}$ are functions of $s=P^2$ and $m_{1,2}$. We then derive

$$\begin{aligned} K_1 \;&=\; \frac{1}{48\,s^2}\big[s^2-2s\,\big(m_1^2+m_2^2\big)+\big(m_1^2-m_2^2\big)^2\big]^2 \,, \\ K_2 \;&=\; \frac{-1}{24\,s^3}\big[s-\big(m_1+m_2\big)^2\big]\big[s-\big(m_1-m_2\big)^2\big]\big[3s^2+s\,\big(4m_1^2-6m_2^2\big)+3\big(m_1^2-m_2^2\big)^2\big] \,, \\ K_3 \;&=\; \frac{1}{s^4}\big\{s^4+s^2\big(m_1^4-6m_1^2m_2^2+6m_2^4\big)+s\big(m_1^2-4m_2^2\big)\big[s^2+\big(m_1^2-m_2^2\big)^2\big]+\big(m_1^2-m_2^2\big)^4\big\} \,, \end{aligned} \tag{B3}$$

$$\begin{aligned} L_1 \;&=\; \frac{-1}{48\,s^2}\big[s^2-2s\,\big(m_1^2+m_2^2\big)+\big(m_1^2-m_2^2\big)^2\big]^2 \,, \\ L_2 \;&=\; \frac{-1}{24\,s^3}\big[s-\big(m_1+m_2\big)^2\big]\big[s-\big(m_1-m_2\big)^2\big]\big[2s^2+s\,\big(m_1^2+m_2^2\big)-3\big(m_1^2-m_2^2\big)^2\big] \,, \end{aligned}$$

[1] Some other examples of the use of Schouten's identity can be found in Ref. [9].

$$L_3 = \frac{1}{24\,s^3}\left[s-\left(m_1+m_2\right)^2\right]\left[s-\left(m_1-m_2\right)^2\right]\left[3s^2+s\left(4m_1^2-6m_2^2\right)+3\left(m_1^2-m_2^2\right)^2\right],$$

$$L_4 = \frac{1}{4\,s^4}\Big[s^4+s^3\left(m_1^2+m_2^2\right)+s^2\left(m_1^4+4m_1^2m_2^2-9m_2^4\right) + s\left(m_1^2+11m_2^2\right)\left(m_1^2-m_2^2\right)^2-4\left(m_1^2-m_2^2\right)^4\Big]. \qquad \text{(B4)}$$

Our formula above for $K^{\mu\nu\rho\sigma}$ $\left(L^{\mu\nu\rho\sigma}\right)$ agrees (disagrees) with that found in Ref. [3]. In obtaining Eq. (22), we set $m_1 = m_2 = m_\pi$.

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